# Theory of random matrices with strong level confinement: Orthogonal polynomial approach

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Strongly non-Gaussian ensembles of large random matrices possessing unitary symmetry and logarithmic level repulsion are studied both in the presence and the absence of a hard edge in their energy spectra. Employing a theory of polynomials orthogonal with respect to exponential weights we calculate with an asymptotic accuracy the two-point kernel over all distance scale, and show that in the limit of large dimensions of random matrices the properly rescaled local eigenvalue correlations are independent of level confinement while global smoothed connected correlations depend on confinement potential only through the end points of the spectrum. We also obtain the exact expressions for density of levels, one- and two-point Green's functions, and prove that a universal local relationship exists for the suitably normalized and rescaled connected two-point Green's function. The connection between the structure of the Szegö function entering strong polynomial asymptotics and mean-field equation is traced. [S1063-651X(96)00807-0]

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## I. INTRODUCTION

Statistical properties of complex physical systems can successfully be investigated within the framework of the random-matrix theory (RMT) [1]. It turned out to be quite general and a powerful phenomenological approach to a description of the various phenomena in such diverse fields as two-dimensional gravity [2], quantum chaos [3], complex nuclei [4], and mesoscopic physics [5].

In all the realms mentioned above the physical systems can be described with the help of different matrix models whose structures depend on physical properties of the systems involved. In the applications of the RMT to the complex quantum-mechanical objects the real Hamiltonian is rather intricate to be handled or simply unknown. In such situations the integration of the exact equations is replaced by the study of the joint distribution function $P[\mathbf{H}]$ of the matrix elements of the Hamiltonian $\mathbf{H}$. If there is not preferential basis in the space of matrix elements (i.e., the system in question is "as random as possible," and equal weight is given to all kinds of interactions) one has to require $P[\mathbf{H}]d[\mathbf{H}]$ to be invariant under similarity transformation $\mathbf{H}\to\mathcal{R}^{-1}\mathbf{H}\mathcal{R}$, with $\mathcal{R}$ being orthogonal, unitary, or a symplectic $n\times n$ matrix reflecting the fundamental symmetry of the underlying Hamiltonian. The general form of $P[\mathbf{H}]$ compatible with invariance requirement is

$$P[\mathbf{H}]=Z^{-1}\exp\{-\mathrm{tr}V[\mathbf{H}]\}, \tag{1}$$

with arbitrary $V[\mathbf{H}]$ providing existence of the partition function $Z$. Introducing the matrix $\mathcal{S}_\beta$ that diagonalizes the Hamiltonian $\mathbf{H}$, $\mathbf{H}=\mathcal{S}_\beta^{-1}\mathbf{X}\mathcal{S}_\beta$, and carrying out the integration over the orthogonal ($\beta=1$), unitary ($\beta=2$), or symplectic ($\beta=4$) group $d\mu(\mathcal{S}_\beta)$ in the construction $P[\mathbf{H}]d[\mathbf{H}]$, one obtains the famous expression for the joint probability density function of the eigenvalues $\{x\}$ of the matrix $\mathbf{H}$

$$P(\{x\})=Z^{-1}\exp\left\{-\beta\left[\sum_i V(x_i)-\sum_{i<j}\ln|x_i-x_j|\right]\right\}. \tag{2}$$

The level repulsion described by the logarithmic term is originated from the Jacobian $\Pi_{i<j}|x_i-x_j|^\beta$ arising when passing from the integration over independent elements $H_{ij}$ of the Hamiltonian $\mathbf{H}$ to the integration over a smaller space of its $n$ eigenvalues $\{x\}$. The confinement potential $V(x)$, which determines (together with the logarithmic law of level repulsion) the mean-level density, contains information about correlations between the different matrix elements of a random Hamiltonian $\mathbf{H}$. [Note that parameter $\beta$ is factored out from $V[\mathbf{H}]$ in Eq. (1) to fix the density of levels in the random-matrix ensembles with the same confinement potential but with different underlying symmetries.]

In the matrix formulation given above the eigenvalues $\{x\}$ of the Hamiltonian $\mathbf{H}$ run from $-\infty$ to $+\infty$. Formally, the same matrix model Eq. (2) appears in the so-called maximum entropy models constructed to describe the transport properties of mesoscopic systems. In this case there is an additional positivity constraint on $\{x\}$, $x\geqslant 0$, that directly follows from the unitarity of the scattering matrix [5,6] and introduces the hard edge into the eigenvalue spectrum.

In the unitary case ($\beta=2$), which applies to the physical systems with broken time-reversal symmetry, the structure of Eq. (2) allows one to represent *exactly* all the global and local statistical characteristics of the physical system, such as the averaged density of levels, $n$-point correlation functions, level-spacing distribution function, etc., in terms of the polynomials orthogonal with respect to the weight function $w(x)=\exp\{-2V(x)\}$ on the whole real axis $\mathbf{R}$ (or on $\mathbf{R}^+$ if there is a hard edge in the eigenvalue spectrum). [Otherwise, when $\beta=1$ or $\beta=4$ more complicated sets of skew orthogonal polynomials should be introduced [7].]

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Analytical calculation of the corresponding set of orthogonal polynomials is a nontrivial problem. However, if the elements $H_{ij}$ of the random-matrix $\mathbf{H}$ are believed to be statistically independent from each other, one obtains the quadratic confinement potential $V(x) \sim x^2$ [8] leading to the Gaussian invariant ensembles of random matrices. In such a case there are significant mathematical simplifications allowing one to solve the matrix model Eq. (1) completely [1].

From the very beginning it was understood [9] that a requirement of the statistical independence of the matrix elements $H_{ij}$ is not motivated by the first principles, and, therefore, several attempts were undertaken to elucidate an influence of a particular form of confinement potential on the predictions of the random-matrix theory developed for Gaussian ensembles.

Two essentially different lines of inquiries of this problem can be distinguished. The first line lies in the framework of the polynomial approach, while a second one consists of the developing of a number of approximate methods. The mean-field approximation proposed by Dyson [10] allows us to calculate density of levels in a random-matrix ensemble. This approach being combined with the method of the functional derivative of Beenakker [11,12] makes it possible to compute global (smoothed) eigenvalue correlations in large random matrices. Smoothed correlations can also be obtained by the diagrammatic approach of Brézin and Zee [13] and by invoking the linear response arguments and macroscopic electrostatics [14]. We stress that all the methods mentioned above allow us to study correlations only in the *long-range regime* and, in this sense, they are less informative as compared with the method of orthogonal polynomials [1]. It is worth pointing out the supersymmetry formalism [15], recently developed for matrix model Eq. (1) with the non-Gaussian probability distribution function $P[\mathbf{H}]$, which is exceptional in that it allows us to investigate *local* eigenvalue correlations and represents a powerful alternative approach to the classical method of orthogonal polynomials.

In the framework of the polynomial approach there was a number of studies to go beyond the Gaussian distribution $P[\mathbf{H}]$. In Refs. [16–18] it was found out that the unitary random-matrix ensembles associated with classical orthogonal polynomials exhibit Wigner-Dyson level statistics (for corresponding ensembles with orthogonal and symplectic symmetry see Ref. [19]). Non-Gaussian unitary random-matrix ensembles associated with (symmetric) strong confinement potentials $V(x)=x^2+\gamma x^4$ and $V(x)=\sum_{n=1}^{n=p} a_n x^{2n}$ were treated in Refs. [7] and [20], respectively. (We note that both potentials mentioned above are stronger than quadratic, and they do not refer to the maximum entropy models.) As far as these works have been based on different *conjectures* about the functional form of asymptotics of polynomials orthogonal with respect to a non-Gaussian measure, and the problem of the hard edge in the eigenvalue spectrum was out of their scope, the polynomial approach to the basic problems of the random-matrix theory needs further and more rigorous study.

The purpose of the present work is to show that the problem of non-Gaussian ensembles with unitary symmetry can be handled rigorously by the method of orthogonal polynomials. Our treatment is exact (i.e., it does not involve any conjectures) and based on the recent results obtained in the theory of polynomials orthogonal with respect to exponential weights on $\mathbf{R}$. It applies to a very large class of confinement potentials which is much richer than that considered in Refs. [7,20] and allows us also to treat the matrix models with positivity constraints on the eigenvalue spectrum. We concentrate on the calculations of the density of levels, one- and two-point Green's functions, the two-point kernel, and the connected ''density-density'' correlation function over the *all distance scale*. This allows us to resolve the problem of universality for local and global correlations of the random-matrix eigenvalues and to establish a universal local relationship for properly normalized and rescaled connected two-point Green's function. One of the interesting points we would like to stress is that the mean-field approximation, widely used in the theory of random matrices, naturally appears in our treatment without any physical speculations and turns out to be closely allied with the structure of the Szegö function entering strong pointwise asymptotics of orthogonal polynomials.

The paper is organized as follows. Section II contains a short introduction to the theory of polynomials orthogonal with respect to the Freud weights. The asymptotic formula for the orthonormal ''wave function'' that we need in later sections is given there. In Sec. III we calculate the two-point kernel and resolve the problem of universality of level statistics. The density of levels and the one-point Green's function are computed in Sec. IV. Connection between the structure of the Szegö function and the mean-field equation is established there as well. Section V is devoted to the calculation of the two-point connected Green's function; a corresponding universal local expression is given. Section VI contains generalizations of the results obtained in the preceding sections for a wider class of random matrices characterized by an Erdös-type confinement potential. In Sec. VII we present a treatment of the maximum entropy models with a hard edge. Finally, in Sec. VIII we discuss the results obtained.

## II. FREUD-TYPE CONFINEMENT POTENTIALS AND CORRESPONDING ORTHOGONAL POLYNOMIALS

Let us consider a class of symmetric (even) confinement potentials $V(x)$ supported on the whole real axis $x \in (-\infty,+\infty)$ which are *of smooth polynomial growth at infinity and increase there at least as* $|x|^{1+\delta}$ ($\delta$ is an arbitrary small positive number). More precisely, we demand that $V(x)$ and $d^2V/dx^2$ be continuous in $x \in (0,+\infty)$, and $dV/dx > 0$ in the same domain of variable $x$. We also assume that for some $A > 1$ and $B > 1$ the inequality

$$A \leqslant 1 + x \frac{d^2V/dx^2}{dV/dx} \leqslant B \tag{3}$$

holds for $x \in (0,+\infty)$, and also for $x$ positive and large enough

$$x^2 \frac{|d^3V/dx^3|}{dV/dx} \leqslant \text{const.} \tag{4}$$

The class of potentials $V(x)$ satisfying all the above requirements is said to be of the *Freud type* [21]. The typical examples of the Freud potentials are (i) $V(x)=|x|^{\alpha}$ with $\alpha>1$, and (ii) $V(x)=|x|^{\alpha}\ln^{\beta}(\gamma+x^2)$ with $\alpha>1$, $\beta\in\mathbf{R}$, and $\gamma$ large enough.

Now it is convenient to introduce a set of polynomials $P_n(x)$ orthogonal with respect to the Freud (non-Gaussian) measure $d\alpha_{\mathcal{F}}(x)= w_{\mathcal{F}}(x)dx=\exp[-2V(x)]dx$,

$$\int_{-\infty}^{+\infty} P_n(x)P_m(x)d\alpha_{\mathcal{F}}(x)=\delta_{nm}, \tag{5}$$

for which the following basic result was obtained by Lubinsky [21]:

$$\lim_{n\to\infty}\int_{-1}^{+1}d\lambda\left\{\sqrt{a_n}P_n(a_n\lambda) -\left(\frac{2}{\pi}\right)^{1/2}\mathrm{Re}\left[z^nD^{-2}\left(F_n;\frac{1}{z}\right)\right]\right\}^2 w_{\mathcal{F}}(a_n\lambda)=0. \tag{6}$$

Here parametrization $z=e^{i\theta}$ and $\lambda=\cos\theta$ was used.

The Szegö function $D(g;z)$, appearing in Eq. (6), is of fundamental importance in the whole theory of orthogonal polynomials [22], and takes the form

$$D(g;z)=\exp\left(\frac{1}{4\pi}\int_{-\pi}^{+\pi}d\varphi\frac{1+ze^{-i\varphi}}{1-ze^{-i\varphi}}\ln g(\varphi)\right). \tag{7}$$

The first argument of the Szegö function in Eq. (6) is

$$F_n(\varphi)=\exp[-V(a_n\cos\varphi)]|\sin\varphi|^{1/2}, \tag{8}$$

and $a_n$ is the $n$th Mhaskar-Rahmanov-Saff number being the positive root of the integral equation [23]

$$n=\frac{2a_n}{\pi}\int_0^1\frac{\lambda d\lambda}{\sqrt{1-\lambda^2}}\left(\frac{dV}{dx}\right)_{x=a_n\lambda}. \tag{9}$$

(In what follows it will be seen that $a_n$ is none other than the band edge for eigenvalues of the corresponding random-matrix ensemble.)

Equation (6) may be rewritten in a different form passing on to the integration over $x=a_n\lambda$ (so that parametrization $x=a_n\cos\theta$ takes place)

$$\lim_{n\to\infty}\int_{-a_n}^{+a_n}dx\left\{P_n(x)-\left(\frac{2}{\pi a_n}\right)^{1/2}\mathrm{Re}\left[z^nD^{-2}\left(F_n;\frac{1}{z}\right)\right]\right\}^2 w_{\mathcal{F}}(x) =0. \tag{10}$$

Analogously, Eq. (9) reads

$$n=\frac{2}{\pi}\int_0^{a_n}\frac{xdx}{\sqrt{a_n^2-x^2}}\frac{dV}{dx}. \tag{11}$$

Since from Eq. (11) it follows that $\lim_{n\to\infty}a_n\neq 0$, we immediately conclude that the expression in the parentheses of Eq. (10) asymptotically tends to zero as $n\to\infty$ on the interval of integration $|x|<a_n$. If one is not interested in the remainder term, we arrive at the asymptotic formula for orthogonal polynomials of the Freud type:

$$P_n(x)=\sqrt{\frac{2}{\pi a_n}}\mathrm{Re}\left[z^nD^{-2}\left(F_n;\frac{1}{z}\right)\right],\quad x\in(-a_n,+a_n). \tag{12}$$

The Szegö function $D(g;e^{i\theta})$ may be represented as [24]

$$D(g;e^{i\theta})=\sqrt{g(\theta)}\exp[i\Gamma(g;\theta)], \tag{13}$$

where

$$\Gamma(g;\theta)=\frac{1}{4\pi}\int_{-\pi}^{+\pi}d\varphi\cot\left(\frac{\theta-\varphi}{2}\right)[\ln g(\varphi)-\ln g(\theta)]. \tag{14}$$

Making use of the representation of Eqs. (13) and (14), noting that $F_n(-\varphi)=F_n(\varphi)$ and $\Gamma(F_n;-\theta)=-\Gamma(F_n;\theta)$, we obtain

$$D\left(F_n;\frac{1}{z}\right)=\exp\left(-\frac{1}{2}V(a_n\cos\theta)\right)|\sin\theta|^{1/4} \times\exp[-i\Gamma(F_n;\theta)]. \tag{15}$$

Then, Eqs. (12) and (15) yield

$$P_n(a_n\cos\theta)=\sqrt{2/\pi a_n}\frac{\exp[V(a_n\cos\theta)]}{|\sin\theta|^{1/2}}\cos[n\theta+\Gamma(F_n^2;\theta)], \tag{16}$$

where

$$\begin{aligned}\Gamma(F_n^2;\theta)&=\frac{1}{4\pi}\int_{-\pi}^{+\pi}d\varphi\cot\left(\frac{\theta-\varphi}{2}\right)[\ln F_n^2(\varphi)-\ln F_n^2(\theta)]\\ &=\frac{1}{4\pi}\int_0^{\pi}d\varphi[\ln F_n^2(\varphi)-\ln F_n^2(\theta)]\\ &\quad\times\left\{\cot\left(\frac{\theta-\varphi}{2}\right)+\cot\left(\frac{\theta+\varphi}{2}\right)\right\}\\ &=\frac{1}{2\pi}\int_0^{\pi}d\varphi[\ln F_n^2(\varphi)-\ln F_n^2(\theta)]\frac{\sin\theta}{\cos\varphi-\cos\theta}.\end{aligned} \tag{17}$$

Introducing the new variable of integration $\xi=a_n\cos\varphi$ and using parametrization $x=a_n\cos\theta$ ($|x|<a_n$), we get

$$\begin{aligned}\gamma_n(x)&=\Gamma(F_n^2;\theta)|_{x=a_n\cos\theta}\\ &=\frac{1}{2\pi}\int_{-a_n}^{+a_n}d\xi\frac{(a_n^2-x^2)^{1/2}}{(a_n^2-\xi^2)^{1/2}}\frac{h(\xi)-h(x)}{\xi-x},\end{aligned} \tag{18}$$

with

$$h(\xi)=-2V(\xi)+\frac{1}{2}\ln\left[1-\left(\frac{\xi}{a_n}\right)^2\right]. \tag{19}$$

Since for $|x|<a_n$

$$\mathrm{P}\int_{-a_n}^{+a_n}\frac{d\xi}{(\xi-x)(a_n^2-\xi^2)^{1/2}}=0 \tag{20}$$

(here P stands for principal value of an integral), Eq. (18) can be rewritten in the form

$$\gamma_n(x) = \frac{1}{2\pi} \mathrm{P} \int_{-a_n}^{+a_n} d\xi \frac{(a_n^2 - x^2)^{1/2}}{(a_n^2 - \xi^2)^{1/2}} \frac{h(\xi)}{\xi - x}. \tag{21}$$

Then we obtain the following asymptotic formula for the orthonormal ''wave functions'' $\psi_n(x) = P_n(x)\exp[-V(x)]$ that we need in what follows:

$$\psi_n(x) = \sqrt{2/\pi a_n} \left[1 - \left(\frac{x}{a_n}\right)^2\right]^{-1/4} \cos\left[n \arccos\left(\frac{x}{a_n}\right) + \gamma_n(x)\right]. \tag{22}$$

We remind you that Eq. (22) is valid for $|x| < a_n$ in the limit $n \to \infty$.

## III. TWO-POINT KERNEL AND UNIVERSAL EIGENVALUE CORRELATIONS

The two-point kernel allowing us to calculate all the global and local characteristics for the random-matrix ensembles is determined as [1]

$$K_n(x,y) = \frac{k_{n-1}}{k_n} \frac{\psi_n(y)\psi_{n-1}(x) - \psi_n(x)\psi_{n-1}(y)}{y - x}, \tag{23}$$

where $k_n$ is the leading coefficient of the orthogonal polynomial $P_n(x)$. Substitution of Eq. (22) into Eq. (23) yields in the large-$n$ limit

$$K_n(x,y) = \frac{2}{\pi a_n} \frac{k_{n-1}}{k_n} \frac{1}{y-x} \left\{\left[1 - \left(\frac{x}{a_n}\right)^2\right] \times \left[1 - \left(\frac{y}{a_n}\right)^2\right]\right\}^{-1/4} [\cos\Phi_{n-1}(x)\cos\Phi_n(y) - \cos\Phi_{n-1}(y)\cos\Phi_n(x)], \tag{24}$$

where

$$\Phi_n(x) = \gamma_n(x) + n \arccos\left(\frac{x}{a_n}\right). \tag{25}$$

In Eq. (24) the fact was used that $\lim_{n\to\infty}(a_{n-1}/a_n) = 1$. Really, as was noted in the preceding section, the Freud-type potentials exhibit a polynomial growth at infinity. Supposing that at large positive $x$ potential $V(x)$ roughly behaves as $x^\rho$ ($\rho > 1$) we immediately obtain the estimate [see Eq. (11)] $a_n \to n^{1/\rho}$ as $n \to \infty$. Then, obviously, $\lim_{n\to\infty}(a_{n-1}/a_n) = 1$. Taking into account this limit and carrying out the changing of the integration variable $\xi' = \xi a_n / a_{n-1}$ in Eq. (21) we easily obtain that in the large-$n$ limit $\gamma_{n-1}(x) = \gamma_n(x)$, and as a consequence

$$\Phi_{n-1}(x) = \Phi_n(x) - \arccos\left(\frac{x}{a_n}\right). \tag{26}$$

Now Eqs. (24) and (26) give us

$$K_n(x,y) = \frac{1}{\pi(y-x)} \left\{\left[1 - \left(\frac{x}{a_n}\right)^2\right]\left[1 - \left(\frac{y}{a_n}\right)^2\right]\right\}^{-1/4} \left\{\cos\Phi_n(x)\cos\Phi_n(y)\frac{x-y}{a_n} - \sin\Phi_n(y)\cos\Phi_n(x)\left[1 - \left(\frac{y}{a_n}\right)^2\right]^{1/2} + \sin\Phi_n(x)\cos\Phi_n(y)\left[1 - \left(\frac{x}{a_n}\right)^2\right]^{1/2}\right\}. \tag{27}$$

When deriving we have used the identity $\lim_{n\to\infty} k_{n-1}/k_n a_n = 1/2$ proved in Ref. [25]. We stress that Eq. (27) is valid for arbitrary $x$ and $y$ lying within the band $(-a_n, +a_n)$.

Equation (27) allows us to determine smoothed (over the rapid oscillations) connected correlations $\nu_c(x,y)$ of the density of eigenvalues $\nu_n(x)$ [12,20],

$$\nu_c(x,y) = \overline{\langle \nu_n(x)\nu_n(y)\rangle} - \overline{\langle \nu_n(x)\rangle\langle \nu_n(y)\rangle} = -\overline{K_n^2(x,y)}, \quad x \neq y \tag{28}$$

by averaging over intervals $|\Delta x| \ll a_n$ and $|\Delta y| \ll a_n$ but still containing many eigenlevels. Direct calculations yield the simple universal relationship

$$\nu_c(x,y) = -\frac{1}{2\pi^2} \frac{a_n^2 - xy}{(x-y)^2 (a_n^2 - x^2)^{1/2} (a_n^2 - y^2)^{1/2}}, \quad x \neq y \tag{29}$$

with dependence on the potential $V(x)$ only through the end point $a_n$ of the spectrum.

Now we turn to the local properties of the two-point kernel. Assuming that in Eq. (27) $|x-y| \ll a_n$ and both $x$ and $y$ stay away from the (soft) band edge $a_n$ we obtain

$$K_n(x,y) = \frac{\sin[\Phi_n(x) - \Phi_n(y)]}{\pi(y-x)}, \tag{30}$$

where $\Phi_n(x)$ is defined by Eq. (25). This two-point kernel may be rewritten in locally universal form. Taking into account the integral representation

$$\Phi_n(x) = \frac{1}{2}\arccos\left(\frac{x}{a_n}\right) - \pi \int_0^x \omega_{a_n}(\xi) d\xi + \frac{\pi}{4}(2n-1), \tag{31}$$

$$\omega_{a_n}(x) = \frac{2}{\pi^2} \mathrm{P} \int_0^{a_n} \frac{\xi d\xi}{\xi^2 - x^2} \frac{dV}{d\xi} \frac{(a_n^2 - x^2)^{1/2}}{(a_n^2 - \xi^2)^{1/2}}, \tag{32}$$

proved in the Appendix, we see that Eq. (30) may be rewritten as

$$K_n(x,y)=\frac{\sin[\pi\int_x^y\omega_{a_n}(\xi)d\xi]}{\pi(y-x)}. \quad (33)$$

The characteristic scale of the changing of $\omega_{a_n}(\xi)$ is $(\omega_{a_n}^{-1}d\omega_{a_n}/d\xi)^{-1}\sim a_n$, so that for $|x-y|\ll a_n$ [that has been supposed in Eq. (30)] Eq. (33) is reduced to the universal form

$$K_n(x,y)=\frac{\sin[\pi\overline{\nu_n}(y-x)]}{\pi(y-x)}, \quad (34)$$

with $\overline{\nu_n}=\omega_{a_n}\frac{1}{2}(x+y)$ playing the role of the local density of levels. Correspondingly, the local two-level cluster function being rewritten in rescaled variables $s$ and $s'$

$$Y_2(s,s')=\left(\frac{K_n^2(x,y)}{\langle\nu_n(x)\rangle\langle\nu_n(y)\rangle}\right)_{\substack{x=x(s)\\ y=y(s')}}=\frac{\sin^2[\pi(s-s')]}{[\pi(s-s')]^2} \quad (35)$$

proves the universal Wigner-Dyson level statistics in the unitary random-matrix ensemble with Freud-type confinement potentials (here $s=\overline{\nu_n}x$ and $s'=\overline{\nu_n}y$ are the eigenvalues measured in the local mean-level spacing).

## IV. DENSITY OF LEVELS AND ONE-POINT GREEN'S FUNCTION

The expression for density of levels is defined as

$$\langle\nu_n(x)\rangle=\langle\mathrm{tr}\,\delta(x-\mathbf{H})\rangle=K_n(x,x), \quad (36)$$

immediately follows from Eq. (30):

$$\langle\nu_n(x)\rangle=-\frac{1}{\pi}\frac{d\Phi_n}{dx}=\frac{1}{\pi}\left(\frac{n}{(a_n^2-x^2)^{1/2}}-\frac{d\gamma_n}{dx}\right) \quad (37)$$

[see Eq. (25)]. Using Eqs. (13) and (18), and the parametrization $x=a_n\cos\theta$, we obtain the formula

$$\langle\nu_n(x=a_n\cos\theta)\rangle =\frac{1}{\pi a_n\sin\theta}\frac{d}{d\theta}[\arg D(e^{-2V(a_n\cos\varphi)}|\sin\varphi|;e^{i\theta})+n\theta], \quad (38)$$

which establishes the connection between the density of levels in the random-matrix ensemble with the Freud-type confinement potential and the Szegö function for the corresponding set of orthogonal polynomials Eq. (7).

Another representation of the level density can be obtained from Eqs. (34) and (32):

$$\langle\nu_n(x)\rangle=\frac{2}{\pi^2}\,\mathrm{P}\int_0^{a_n}\frac{\xi d\xi}{\xi^2-x^2}\frac{dV}{d\xi}\frac{(a_n^2-x^2)^{1/2}}{(a_n^2-\xi^2)^{1/2}}. \quad (39)$$

This formula is rather interesting and deserves more attention. Considering this expression as an equation for $dV/dx$ one can resolve it invoking the theory of integral equations with a Cauchy kernel [26]:

$$\mathrm{P}\int_{-a_n}^{+a_n}\frac{\langle\nu_n(x')\rangle}{x-x'}dx'=\frac{dV}{dx}. \quad (40)$$

Thus one can think that density of levels is a solution of the integral equation

$$V(x)=\int_{-a_n}^{+a_n}dx'\langle\nu_n(x')\rangle\ln|x-x'|+\mu, \quad (41)$$

with $\mu$ being the "chemical potential." It is no more than the famous mean-field equation which, in our treatment, finally follows from the asymptotic formula Eq. (12) for the orthogonal polynomials. Quite surprisingly, the Szegö function Eq. (7) turns out to be closely related to the mean-field approximation by Dyson [10].

Now we can easily calculate the one-point Green's function

$$G^p(x)=\left\langle\mathrm{tr}\frac{1}{x-\mathbf{H}+ip0}\right\rangle =\int_{-a_n}^{+a_n}d\xi\frac{1}{x-\xi+ip0}\langle\mathrm{tr}\,\delta(\xi-\mathbf{H})\rangle, \quad (42)$$

where $p=\pm1$. The last integral can be rewritten as

$$G^p(x)=\mathrm{P}\int_{-a_n}^{+a_n}d\xi\frac{\langle\nu_n(\xi)\rangle}{x-\xi}-i\pi p\langle\nu_n(x)\rangle, \quad (43)$$

whence we obtain by means of Eqs. (39) and (40)

$$G^p(x)=\frac{dV}{dx}-\frac{2ip}{\pi}\,\mathrm{P}\int_0^{a_n}\frac{\xi d\xi}{\xi^2-x^2}\frac{dV}{d\xi}\frac{(a_n^2-x^2)^{1/2}}{(a_n^2-\xi^2)^{1/2}}. \quad (44)$$

We would like to stress that both Eqs. (39) and (44) have been obtained within the framework of the theory of polynomials orthogonal with respect to the Freud measure. This comment equally pertains to the mean-field Eq. (41).

## V. TWO-POINT CONNECTED GREEN'S FUNCTION

The two-point connected Green's function is defined as

$$G_c^{pp'}(x,x')=\left\langle\mathrm{tr}\frac{1}{x_p-\mathbf{H}}\mathrm{tr}\frac{1}{x'_{p'}-\mathbf{H}}\right\rangle-\left\langle\mathrm{tr}\frac{1}{x_p-\mathbf{H}}\right\rangle \times\left\langle\mathrm{tr}\frac{1}{x'_{p'}-\mathbf{H}}\right\rangle, \quad (45)$$

where $x_p=x+ip0$ and $x'_{p'}=x'+ip'0$ $(p,p'=\pm1)$. It can be rewritten in an integral form

$$G_c^{pp'}(x,x')=\int_{-a_n}^{+a_n}\int_{-a_n}^{+a_n}\frac{d\xi d\eta}{(x_p-\xi)(x'_{p'}-\eta)}[\langle\nu_n(\xi)\nu_n(\eta)\rangle -\langle\nu_n(\xi)\rangle\langle\nu_n(\eta)\rangle]. \quad (46)$$

Recognizing that the quantity in parentheses is $\langle\nu_n(\xi)\nu_n(\eta)\rangle_c=\langle\nu_n(\xi)\rangle\delta(\xi-\eta)-K_n^2(\xi,\eta)$, we obtain the formula

$$G_c^{pp'}(x,x') = \int_{-a_n}^{+a_n} \frac{d\xi \langle \nu_n(\xi)\rangle}{(x_p-\xi)(x'_{p'}-\xi)} + \pi^2 pp' K_n^2(x,x') + i\pi[p\Lambda(x,x') + p'\Lambda(x',x)] - \lambda(x,x'), \tag{47}$$

where the following notations were used:

$$\Lambda(x,x') = \mathrm{P}\int_{-a_n}^{+a_n} d\xi \frac{K_n^2(x,\xi)}{x'-\xi}, \tag{48}$$

$$\lambda(x,x') = \mathrm{P}\int_{-a_n}^{+a_n} d\xi \frac{\Lambda(\xi,x')}{x-\xi}. \tag{49}$$

The two-point kernel $K_n(x,x')$ entering Eqs. (47) and (48) is determined by Eq. (27).

### A. Smoothed connected two-point Green's function

Let us consider the first integral Eq. (48). Substituting Eq. (27) into Eq. (48) and taking into account that terms of the type $\sin\Phi_n(\xi)$, $\cos\Phi_n(\xi)$, and $\sin\Phi_n(\xi)\cos\Phi_n(\xi)$ oscillate rapidly and, therefore, do not contribute into the integral over $\xi$ in the leading order in $n \gg 1$, we have after some rearrangements

$$\Lambda(x,x') = \frac{1}{2\pi^2}\frac{1}{\sqrt{1-x^2/a_n^2}} \times \mathrm{P}\int_{-a_n}^{+a_n} \frac{d\xi}{(x'-\xi)(x-\xi)^2}\frac{1}{\sqrt{1-\xi^2/a_n^2}}\left(1-\frac{x\xi}{a_n^2}\right), \tag{50}$$

provided $x \neq x'$. Formally, this integral is divergent thanks to the double pole of the integrand $\propto (x-\xi)^{-2}$. It is easy to see that this singularity is rather artificial and connected with the fact that the condition $x \neq \xi$ was supposed to be fulfilled when neglecting rapid oscillations in $\xi$ in the integrand of Eq. (48). This is the reason why the integrand in Eq. (50) displays incorrect behavior in the vicinity $x=\xi$. Actually, as can be verified, the integrand is finite for $x=\xi$, and the corresponding integral is convergent. Moreover, direct comparison of Eq. (50) with results [12] shows that the equation in question can be rewritten in the form

$$\Lambda(x,x') = \mathrm{P}\int_{-a_n}^{+a_n} \frac{d\xi}{x'-\xi} T_2(\xi,x), \tag{51}$$

where

$$T_2(\xi,x) = K_2(\xi,x) + \langle \nu_n(\xi)\rangle \delta(\xi - x) \tag{52}$$

is the two-level cluster function, and

$$K_2(\xi,x) = \frac{1}{2}\frac{\delta\langle \nu_n(\xi)\rangle}{\delta V(x)} \tag{53}$$

is the two-point correlation function (the notations of Ref. [12] have been used). Then, taking into account Eqs. (51), (52), and (49), we obtain from Eq. (47) after some transformations

$$\overline{G_c^{pp'}(x,x')} = \pi^2 pp' \overline{K_n^2(x,x')} + i\pi\left[p\mathrm{P}\int_{-a_n}^{+a_n}\frac{d\xi}{x'-\xi}K_2(\xi,x) + p'\,\mathrm{P}\int_{-a_n}^{+a_n}\frac{d\xi}{x-\xi}K_2(\xi,x')\right] - \mathrm{PP}\int_{-a_n}^{+a_n}\int_{-a_n}^{+a_n}\frac{d\xi d\eta}{(x-\xi)(x'-\eta)}K_2(\xi,\eta). \tag{54}$$

Now we only have to calculate the integrals containing $K_2$. The most proper way is to invoke the integral equation [12]

$$\mathrm{P}\int_{-a_n}^{+a_n}\frac{d\xi}{x-\xi}\delta\langle \nu_n(\xi)\rangle = \frac{d}{dx}\delta V(x) \tag{55}$$

and definition Eq. (53). Since Eqs. (53) and (55) yield identity

$$i\pi\left[p\mathrm{P}\int_{-a_n}^{+a_n}\frac{d\xi}{x'-\xi}K_2(\xi,x) + p'\,\mathrm{P}\int_{-a_n}^{+a_n}\frac{d\xi}{x-\xi}K_2(\xi,x')\right] - \mathrm{PP}\int_{-a_n}^{+a_n}\int_{-a_n}^{+a_n}\frac{d\xi d\eta}{(x-\xi)(x'-\eta)}K_2(\xi,\eta) = -\frac{1}{2}\frac{1}{(x_p-x'_{p'})^2}, \tag{56}$$

we finally arrive at the expression for the two-point connected Green's function

$$\overline{G_c^{pp'}(x,x')} = \frac{1}{2}\left\{pp'\frac{a_n^2-xx'}{(x-x')^2(a_n^2-x^2)^{1/2}(a_n^2-x'^2)^{1/2}} - \frac{1}{(x_p-x'_{p'})^2}\right\}. \tag{57}$$

Here we have used Eqs. (28) and (29). Equation (57) is valid for arbitrary $x \neq x'$ lying within the band $(-a_n, +a_n)$. Universal relationships of this type were obtained in Ref. [20].

### B. Local connected two-point Green's function

In the local regime, when $|x-x'| \ll a_n$, one cannot disregard oscillations of the integrands in Eqs. (48) and (49). Since in this energy scale the density of states $\langle \nu_n(x)\rangle$ is a slowly varying function and the two-point kernel $K_n(x,x')$ is universal Eq. (34) one obtains that [28]

$$\Lambda(x,x') = \frac{\overline{\nu_n}}{x'-x}\left\{1 - \frac{\sin[2\pi\overline{\nu_n}(x'-x)]}{2\pi\overline{\nu_n}(x'-x)}\right\} \tag{58}$$

and

$$\lambda(x,x') = \frac{\sin^2[\pi\overline{\nu_n}(x-x')]}{(x-x')^2}. \tag{59}$$

Then Eqs. (58), (59), and (47) yield

$$G_c^{pp'}(x,x') = \pi^2 \overline{\nu_n} |p-p'| \delta(x-x') + [pp'-1] \frac{\sin^2[\pi \overline{\nu_n}(x-x')]}{(x-x')^2} + i(p'-p) \times \frac{\sin[\pi \overline{\nu_n}(x'-x)] \cos[\pi \overline{\nu_n}(x'-x)]}{(x'-x)^2}. \quad (60)$$

This equation only depends on the local mean-level spacing $\overline{\nu_n}$, and therefore it can be written down in universal form. Introducing the normalized and rescaled two-point connected Green's function

$$g_c^{pp'}(s,s') = \left( \frac{G_c^{pp'}(x,x')}{\langle \nu_n(x) \rangle \langle \nu_n(x') \rangle} \right)_{\substack{x=x(s) \\ x'=x'(s')}}, \quad (61)$$

where $s = \overline{\nu_n} x$ and $s' = \overline{\nu_n} x'$ are the eigenvalues measured in the local mean-level spacing, we obtain the following universal relationship:

$$g_c^{pp'}(s,s') = \pi^2 |p-p'| \delta(s-s') + i(p-p') \frac{\sin[\pi(s-s')]}{(s-s')^2} e^{i\pi(s-s')\text{sign}(p-p')}. \quad (62)$$

Note that an expression of this type was previously obtained in Ref. [29] only for the Gaussian random-matrix ensemble using supersymmetry formalism.

## VI. EXTENSION FOR ERDÖS-TYPE CONFINEMENT POTENTIALS

All the results obtained above are valid for confinement potentials exhibiting smooth polynomial growth at infinity (see Sec. II) but they can be extended for an *Erdös-type* confinement potential which *grows faster than any polynomial at infinity* (see Ref. [27], Ch. 2).

Namely, let $V(x)$ be even and continuous in $x \in (-\infty, +\infty)$, $d^2V/dx^2$ be continuous in $x \in (0,+\infty)$, $dV/dx$ be positive in the same domain of $x$ and continuous at $x=0$. Moreover, let

$$T(x) = 1 + x \frac{d^2V/dx^2}{dV/dx} \quad (63)$$

be positive and increasing in $x \in (0,+\infty)$ with $\lim_{x\to+0} T(x) > 0$ while $\lim_{x\to\infty} T(x) = \infty$, and

$$T(x) = O((dV/dx)^{1/15}) \quad \text{for } x \to \infty, \quad (64)$$

$$\frac{d^2V/dx^2}{dV/dx} \sim \frac{dV/dx}{V(x)} \quad \text{and} \quad \frac{|d^3V/dx^3|}{dV/dx} \leq \text{const} \left( \frac{dV/dx}{V(x)} \right)^2 \quad (65)$$

for $x$ large enough. The class of potentials $V(x)$ satisfying all the above requirements is said to be of the *Erdös type*. The simple examples of Erdös-type confinement potentials are (i) $V(x) = \exp_k(|x|^\alpha)$ with $\alpha > 0$ and $k \geq 1$ (here $\exp_k$ denotes the exponent iterated $k$ times); (ii) $V(x) = \exp[\ln^\alpha(\gamma + x^2)]$ with $\alpha > 1$, and $\gamma$ large enough.

Polynomials orthogonal with respect to the Erdös measure $d\alpha_{\mathcal{E}} = w_{\mathcal{E}}(x)dx = \exp[-2V(x)]dx$ (here $V$ is of Erdös type) have the same asymptotics [27] and, therefore, Eq. (22) remains valid along with all the results obtained in Secs. III, IV, and V.

## VII. MATRIX MODELS WITH POSITIVITY CONSTRAINTS ON EIGENVALUES

In the random-matrix theory of quantum transport [5,6] the matrix model Eq. (2) appears with positivity constraints on eigenvalues $\{x\}$ (maximum entropy models). The constraint $x \geq 0$ is an essential feature of those models that follows directly from the unitarity of the scattering matrix and imposes the presence of the hard edge in the energy spectrum of the matrix model. To our knowledge there is no rigorous treatment of such a matrix model with a strong enough confinement potential $V(x)$ within the method of orthogonal polynomials except for the generalized Laguerre ensembles of random matrices [30].

Below we show how the problems associated with the maximum entropy model can be treated within the polynomial approach in a very general case.

### A. Polynomials orthogonal on $x \geq 0$

Let the confinement potential $V(x)$ be of the Freud or Erdös type defined on the whole real axis $\mathbf{R}$, that is, $V$ is a monotonous function behaving at least as $|x|^{1+\delta}$ $(\delta > 0)$ and growing as or even faster than any polynomial at infinity, and let $P_n(x)$ be a set of polynomials orthogonal on $\mathbf{R}$ with respect to the measure $d\alpha(x) = \exp\{-2V(x)\}dx$ [see Eq. (5)]. Then polynomials

$$S_n(x) = P_{2n}(\sqrt{x}) \quad (66)$$

form a set of polynomials orthogonal on $\mathbf{R}^+$ with the measure [31] $d\alpha_s(x) = \exp\{-2V_s(x)\}dx$,

$$\int_0^\infty S_n(x) S_m(x) d\alpha_s(x) = \delta_{nm}, \quad (67)$$

where the confinement potential

$$V_s(x) = V(\sqrt{x}) + \tfrac{1}{4} \ln x \quad (68)$$

is a monotonous function that behaves at least as $|x|^{\frac{1}{2}+\delta}$ $(\delta > 0)$ and can grow even faster than any polynomial at infinity.

Equation (66) allows us to write down large-$n$ asymptotics for the introduced set of orthogonal polynomials. It is straightforward to get from the results outlined in Sec. II and the Appendix the following asymptotic formula [which is an analogue of Eq. (16)]:

$$S_n(x) = \sqrt{(2/\pi)} \frac{\exp[V_s(x)]}{(xb_n)^{1/4}} \frac{1}{[1-x/b_n]^{1/4}} \cos \tilde{\Phi}_n(x), \quad (69)$$

where $x \in (0, b_n)$, and

$$\tilde{\Phi}_n(x) = \frac{1}{2}\arccos(\sqrt{x/b_n}) + \pi\left(n - \frac{1}{4}\right) - \pi\int_0^x \Omega_{b_n}(\xi)d\xi, \tag{70}$$

$$\Omega_{b_n}(x) = \frac{1}{\pi^2}\,\mathrm{P}\int_0^{b_n}\frac{d\eta}{\eta - x}\,\frac{dV_s}{d\eta}\sqrt{\eta/x}\,\frac{\sqrt{b_n - x}}{\sqrt{b_n - \eta}}. \tag{71}$$

Here the soft band edge $b_n = a_{2n}^2$. The equations obtained above are the starting point of further analysis.

### B. Two-point kernel and universal eigenvalue correlations

The two-point kernel determined by Eq. (23) can be calculated provided ''wave function'' $\psi_n(x) = \exp[-V_s(x)]S_n(x)$. Substitution of Eq. (69) into Eq. (23) yields in the large-$n$ limit

$$K_n(x,y) = \frac{2}{\pi}\,\frac{\tilde{k}_{n-1}}{\tilde{k}_n}\,\frac{1}{(y-x)}\,\frac{1}{(xy)^{1/4}\{[b_n - x][b_n - y]\}^{1/4}} \times[\cos\tilde{\Phi}_{n-1}(x)\cos\tilde{\Phi}_n(y) - \cos\tilde{\Phi}_{n-1}(y)\cos\tilde{\Phi}_n(x)], \tag{72}$$

if $x$ and $y$ lie within the band $(0,b_n)$. If at least one of the arguments in the two-point kernel is negative, it is identically zero (due to the presence of the hard edge). In Eq. (72) $\tilde{k}_n$ stands for the leading coefficient of $S_n(x)$.

Taking into account the large-$n$ identity

$$\tilde{\Phi}_{n-1}(x) = \tilde{\Phi}_n(x) - 2\arccos(\sqrt{x/b_n}), \tag{73}$$

we obtain

$$K_n(x,y) = \frac{4}{\pi}\,\frac{\tilde{k}_{n-1}}{\tilde{k}_n}\,\frac{1}{(y-x)}\,\frac{1}{(xy)^{1/4}\{[b_n - x][b_n - y]\}^{1/4}}\left\{\cos\tilde{\Phi}_n(x)\cos\tilde{\Phi}_n(y)\,\frac{x-y}{b_n} - \sin\tilde{\Phi}_n(y)\cos\tilde{\Phi}_n(x)\left(\frac{y}{b_n}\right)^{1/2}\left(1 - \frac{y}{b_n}\right)^{1/2} + \sin\tilde{\Phi}_n(x)\cos\tilde{\Phi}_n(y)\left(\frac{x}{b_n}\right)^{1/2}\left(1 - \frac{x}{b_n}\right)^{1/2}\right\}. \tag{74}$$

The smoothed (over the rapid oscillations) connected correlator $\nu_c(x,y)$ of the density of eigenvalues Eq. (28)

$$\nu_c(x,y) = -\frac{1}{2\pi^2}\,\frac{b_n(x+y)/2 - xy}{(x-y)^2\sqrt{xy}\sqrt{b_n - x}\sqrt{b_n - y}}, \quad x \neq y \tag{75}$$

manifests dependence on the potential $V_s(x)$ only through the soft edge $b_n$ of the spectrum.

The local properties of the two-point kernel are obtained by assuming that in Eq. (74) $|x-y| \ll b_n$ and both $x$ and $y$ stay away from the hard edge $x=0$ and soft edge $x=b_n$

$$K_n(x,y) = \frac{\sin[\pi\int_x^y \Omega_{b_n}(\xi)d\xi]}{\pi(y-x)}. \tag{76}$$

The characteristic scale of the changing of $\Omega_{b_n}(\xi)$ is of the order of $b_n$, so that for $|x-y| \ll b_n$ Eq. (76) is reduced to the universal form Eq. (34) with $\overline{\nu_n} = \Omega_{b_n}(\frac{1}{2}(x+y))$ playing the role of the local density of levels. Correspondingly, the local two-level cluster function $Y_2(s,s')$ being rewritten in rescaled variables $s$ and $s'$ follows the universal form Eq. (35) that proves the universal Wigner-Dyson level statistics in the bulk of the spectrum for unitary random-matrix ensembles with confinement potentials $V_s(x)$.

### C. Density of levels and one-point Green's function

The density of levels is obtained from Eq. (76) in the limit $y \to x$:

$$\langle \nu_n(x)\rangle = \frac{1}{\pi^2}\,\mathrm{P}\int_0^{b_n}\frac{d\eta}{\eta - x}\,\frac{dV_s}{d\eta}\left(\frac{\eta}{x}\right)^{1/2}\frac{\sqrt{b_n - x}}{\sqrt{b_n - \eta}}. \tag{77}$$

Considering this expression as an equation for $dV_s/dx$ one can resolve it [26] arriving to the mean-field equation by Dyson

$$V_s(x) = \int_0^{b_n} dx'\langle \nu_n(x')\rangle \ln|x - x'| + \mu, \tag{78}$$

where integration runs over $x' \in (0,b_n)$. We once more stress that the mean-field equation is a direct consequence of the point-wise asymptotics for the corresponding orthogonal polynomials $S_n(x)$ which involve the Szegö function as a starting point.

Correspondingly, the one-point Green's function

$$G^p(x) = \frac{dV_s}{dx} - \frac{ip}{\pi}\,\mathrm{P}\int_0^{b_n}\frac{d\eta}{\eta - x}\,\frac{dV_s}{d\eta}\left(\frac{\eta}{x}\right)^{1/2}\frac{\sqrt{b_n - x}}{\sqrt{b_n - \eta}}. \tag{79}$$

### D. Connected two-point Green's function

In the maximum entropy models the smoothed connected two-point Green's function can be calculated in the same way as was done in Sec. V. The only difference is that the integrals in Eqs. (47)–(49) and (54) now run from 0 to $b_n$. Carrying out this integration with the two-point kernel $K_n(x,y)$ given by Eq. (74) we arrive at the universal formula

$$\overline{G_c^{pp'}(x,x')} = \frac{1}{2}\left\{ pp' \frac{(b_n/2)(x+x') - xx'}{(x-x')^2 \sqrt{xx'}\sqrt{b_n - x}\sqrt{b_n - x'}} - \frac{1}{(x_p - x'_{p'})^2} \right\}. \tag{80}$$

In contrast to the smoothed connected two-point Green's function the local one is determined by the same formulas Eqs. (60)–(62) provided $x$ and $y$ are far from both edges.

## VIII. CONCLUSION

We have presented rigorous analytical consideration of the matrix model given by the non-Gaussian distribution function $P(\{x\})$ Eq. (2) with a very general class of confinement potentials $V(x)$ within the framework of the orthogonal-polynomials technique. Our treatment is equally applied to the random-matrix models with the presence and absence of the hard edge in the eigenvalue spectrum. We have calculated with asymptotic accuracy the density of levels, the one-point Green's function, the two-point kernel, the "density-density" correlator, and the two-point Green's function over the all distance scale.

It was established that the two-point correlators in considered random-matrix model possess a high degree of universality. In the absence of the hard edge the universality is observed for a very wide class of monotonous confinement potentials $V(x)$ which behave at least as $|x|^{1+\delta}$ ($\delta > 0$) and *can grow as or even faster than any polynomial at infinity* (the case of the border level confinement when $V(x) \sim |x|$ as $|x| \to \infty$ has been treated in Ref. [32]). In the presence of the hard edge in the eigenvalue spectrum the universality holds for the monotonous confinement potentials $V_s(x)$ which behave at least as $|x|^{1/2+\delta}$ ($\delta > 0$) and *can grow faster than any polynomial at infinity.*

We have shown that in those unitary non-Gaussian random-matrix models the density of levels and the one-point Green's function essentially depend on the measure, i.e., on the explicit form of the confinement potential. In contrast (connected) the two-point characteristics of the spectrum ("density-density" correlator, two-point Green's function) are rather universal. Indeed, we have observed global universality of smoothed two-point connected correlators and local universality of those without smoothing over rapid oscillations. In both cases the correlators were shown to depend on the measure only through the end points of the spectrum (global universality) or through the local density of levels (local universality).

Rigorous polynomial analysis enabled us to recover the results obtained before by different approximate methods and to extend previously known results for a much wider class of random-matrix ensembles with strong confinement potentials irrespective of the presence or absence of a hard edge. We have also established a local universal relationship for the normalized and rescaled connected two-point Green's function $g_c^{pp'}(s,s')$ [see Eq. (62)]. Finally, an interesting and quite surprising intimate connection between the structure of the Szegö function and the mean-field equation that has been revealed in the proposed formalism is worthy of notice.

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## APPENDIX: INTEGRAL REPRESENTATION OF $\Phi_n(x)$

To prove Eq. (31) let us calculate the first derivative of $\gamma_n(x)$ (the calculations are similar to those done in Ref. [27], Ch. 11). From Eq. (21) we obtain

$$\frac{d\gamma_n}{dx} = -\frac{1}{2\pi}\frac{x}{(a_n^2 - x^2)^{1/2}} \mathrm{P}\int_{-a_n}^{+a_n} \frac{h(\xi)d\xi}{(a_n^2-\xi^2)^{1/2}(\xi - x)} + \frac{1}{2\pi}(a_n^2 - x^2)^{1/2}\mathrm{P}\int_{-a_n}^{+a_n}\frac{h(\xi)d\xi}{(a_n^2-\xi^2)^{1/2}(\xi-x)^2}, \tag{A1}$$

whence

$$(a_n^2 - x^2)^{1/2}\frac{d\gamma_n}{dx} = \frac{1}{2\pi}\mathrm{P}\int_{-a_n}^{+a_n}\frac{h(\xi)d\xi}{\xi - x}\left(\frac{\xi}{(a_n^2-\xi^2)^{1/2}} + \frac{(a_n^2-\xi^2)^{1/2}}{\xi - x}\right) = -\frac{1}{2\pi}\mathrm{P}\int_{-a_n}^{+a_n} h(\xi)d\xi \frac{d}{d\xi}\left(\frac{(a_n^2-\xi^2)^{1/2}}{\xi - x}\right). \tag{A2}$$

After integration by parts we have

$$\frac{d\gamma_n}{dx} = \frac{1}{\pi(a_n^2 - x^2)^{1/2}}\mathrm{P}\int_0^{a_n} d\xi \frac{(a_n^2-\xi^2)^{1/2}}{\xi^2 - x^2}\xi\frac{dh}{d\xi}. \tag{A3}$$

Substituting Eq. (19) into Eq. (A3) and using identity

$$\mathrm{P}\int_0^{a_n}\frac{d\xi}{\xi^2 - x^2}\frac{1}{(a_n^2-\xi^2)^{1/2}} = 0 \tag{A4}$$

we obtain

$$\frac{d\gamma_n}{dx} = -\frac{2}{\pi(a_n^2-x^2)^{1/2}}\mathrm{P}\int_0^{a_n} d\xi\frac{(a_n^2-\xi^2)^{1/2}}{\xi^2-x^2}\xi\frac{dV}{d\xi} - \frac{1}{2(a_n^2-x^2)^{1/2}}. \tag{A5}$$

The integral in Eq. (A5) may be handled as follows:

$$\mathrm{P}\int_0^{a_n} d\xi \frac{(a_n^2-\xi^2)^{1/2}}{\xi^2-x^2}\xi\frac{dV}{d\xi} = \mathrm{P}\int_0^{a_n}\frac{\xi d\xi}{\xi^2-x^2}\frac{dV}{d\xi}\frac{(a_n^2-x^2)^{1/2}}{(a_n^2-\xi^2)^{1/2}} - \frac{1}{(a_n^2-x^2)^{1/2}} \times \int_0^{a_n}\frac{\xi d\xi}{(a_n^2-\xi^2)^{1/2}}\frac{dV}{d\xi}. \tag{A6}$$

Bearing in mind Eq. (11) and introducing the function

$$\omega_{a_n}(x)=\frac{2}{\pi^2}\,\mathrm{P}\int_0^{a_n}\frac{\xi d\xi}{\xi^2-x^2}\,\frac{dV}{d\xi}\,\frac{(a_n^2-x^2)^{1/2}}{(a_n^2-\xi^2)^{1/2}} \tag{A7}$$

the derivative $d\gamma_n/dx$ can be rewritten as

$$\frac{d\gamma_n}{dx}=-\pi\omega_{a_n}(x)+\left(n-\frac{1}{2}\right)\frac{1}{(a_n^2-x^2)^{1/2}}. \tag{A8}$$

Further, noting from Eq. (21) that $\gamma_n(0)=0$, we obtain the integral representation

$$\gamma_n(x)=-\pi\int_0^x\omega_{a_n}(\xi)d\xi+\left(n-\frac{1}{2}\right)\arcsin\left(\frac{x}{a_n}\right), \tag{A9}$$

or, equivalently [see Eq. (25)],

$$\Phi_n(x)=\frac{1}{2}\arccos\left(\frac{x}{a_n}\right)-\pi\int_0^x\omega_{a_n}(\xi)d\xi+\frac{\pi}{4}(2n-1). \tag{A10}$$

[1] M. L. Mehta, *Random Matrices* (Academic, Boston, 1991).
[2] P. Di Franceso, P. Ginsparg, and J. Zinn-Justin, Phys. Rep. **254**, 1 (1995).
[3] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
[4] T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[5] See relevant papers in *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (North-Holland, Amsterdam, 1991).
[6] K. Slevin and T. Nagao, Int. J. Mod. Phys. **9**, 103 (1995).
[7] G. Mahoux and M. L. Mehta, J. Phys. (France) **1**, 1093 (1991).
[8] C. E. Porter and N. Rosenzweig, Phys. Rev. **120**, 1698 (1960).
[9] F. J. Dyson, J. Math. Phys. **3**, 140 (1962).
[10] F. J. Dyson, J. Math. Phys. **13**, 90 (1972).
[11] C. W. J. Beenakker, Phys. Rev. Lett. **70**, 1155 (1993); Phys. Rev. B **47**, 15 763 (1993).
[12] C. W. J. Beenakker, Nucl. Phys. B[FS] **422**, 515 (1994).
[13] E. Brézin and A. Zee, Phys. Rev. E **49**, 2588 (1994).
[14] B. Jancovici and P. J. Forrester, Phys. Rev. B **50**, 14 599 (1994).
[15] G. Hackenbroich and H. A. Weidenmüller, Phys. Rev. Lett. **74**, 4118 (1995).
[16] D. Fox and P. B. Kahn, Phys. Rev. B **134**, 1151 (1964).
[17] H. S. Leff, J. Math. Phys. **5**, 763 (1964).
[18] B. V. Bronk, J. Math. Phys. **6**, 228 (1965).
[19] T. Nagao and M. Wadati, J. Phys. Soc. Jpn. **60**, 3298 (1991).
[20] E. Brézin and A. Zee, Nucl. Phys. B[FS] **402**, 613 (1993).
[21] D. S. Lubinsky, Acta Appl. Math. **33**, 121 (1993).
[22] G. Szegö, Math. Annalen **84**, 232 (1921).
[23] H. N. Mhaskar and E. B. Saff, Trans. Am. Math. Soc. **285**, 203 (1984).
[24] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, Providence, 1967), Vol. XXIII, Ch. 10.
[25] D. S. Lubinsky, H. N. Mhaskar, and E. B. Saff, Constr. Appr. 4, **65** (1988).
[26] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Ungar, New York, 1961), Vol. I, Ch. 4.
[27] D. S. Lubinsky, *Strong Asymptotics for Extremal Errors and Polynomials Associated with Erdös-type Weights*, Pitman Research Notes in Mathematics 202 (Longman, Harlow, Essex, 1989).
[28] Bateman Manuscript Project (Director: A. Erdélyi), *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).
[29] J. A. Zuk, Electronic Archive, COND-MAT/9412060 (http://xxx.LANL.GOV)
[30] T. Nagao and K. Slevin, J. Math. Phys. **34**, 2075 (1993); **34**, 2317 (1993).
[31] T. S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).
[32] V. Freilikher, E. Kanzieper, and I. Yurkevich, Phys. Rev. E **53**, 2200 (1996),